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ON THE COMPLETENESS OF A SYSTEM OF ELEMENTARY SOLUTIONS OF THE BIHARMONIC EQUATION IN A SEMI-STRIP

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The problem of completeness of a system of elementary solutions in the space of biharmonic functions with finite energy is investigated. The problem arises during the study of infinite systems of linear algebraic equations in the asymptotic theory of plates. Actually a more general theory is developed here, including e.g. orthotropic and transversely inhomogeneous plates. The problem of existence of elementary solutions is solved at the same time. The results concerning the completeness obtained here are independent of the form of the boundary conditions at the end and can, consequently, be applied to a fairly wide class of elliptic boundary value problems which, in particular, appear in the theory of thick plates.

Before the problems of completeness are discussed, we study the problem of traces for the solution of a certain elliptic equation in a semi-cylinder. The necessary and sufficient conditions are formulated for the boundary values which ensure that the solution belongs to the energy space.

As we know, the stress-strain state of a plate can be separated into the internal state and the boundary layer [1 – 4]. Construction of the boundary layer involves consecutive solutions of the plane problems of the theory of elasticity in a semi-strip. Papkovich [5] and others reduce the boundary value problem of the theory of elasticity in the semi-strip $x > 0$, $|y| \leqslant 1$ to finding a biharmonic Airy function, which is sought in the form

$$u = \sum_{\operatorname{Im} \sigma_k > 0} C_k \varphi_k(y) e^{i\sigma_k x}$$

where φ_k are the Papkovich functions [5, 6], σ_k denote the eigenvalues of a certain nonselfconjugate boundary value problem and C_k are unknown constants. In this connection the author of [6] poses the problem of representing a pair of functions f_1 and f_2 in the form

$$\sum_{k=1}^{\infty} C_k P_k \varphi_k = f_1, \qquad \sum_{k=1}^{\infty} C_k Q_k \varphi_k = f_2 \tag{0.1}$$

where P_k and Q_k are differential operators determined by the boundary conditions at $x = 0$. Certain sufficient conditions for the uniform convergence of the series (0.1) are given in [7, 8] for the cases when the coefficients C_k can be obtained in the explicit form using the generalized conditions of orthogonality.

Vorovich has shown in [9] that the present problem is related to the problem of n-tuple completeness discussed by Keldysh in [10], and suggested a novel method (realized in [11]) of investigating the expansions (0.1) based on direct study of the initial boundary value problem. In [11] the coefficients C_k are uniquely defined by the boundary values of the biharmonic function and its derivatives. Thus the completeness and the basic properties of the elementary solutions are both found to be closely connected with the differential properties of the biharmonic function in a region with corners (the problem of traces).

Amongst the recent investigations we note [12] where a theorem was announced concerning n-tuple completeness in the space L_2 of a part of the eigenvectors and adjoint vectors belonging to the operator bundle generated by a certain boundary problem for an elliptic equation in a semi-strip.

The problem of traces for a two-dimensional region with a piecewise smooth boundary was discussed in [13], while [14] dealt with the differential properties of solutions of the general elliptic equations in regions with conical and corner points. Certain new results pertaining to the biharmonic equation are given in [15], and [16 – 18] deal with the behavior of the solutions of the problems of the theory of elasticity near the singular points on the boundary.

1. Let $\Omega = S \times [0, \infty)$ be an n-dimensional semi-cylinder of cross section S and the axis $x_n = x$, Γ its side surface, γ the boundary of S, and S_0 the end surface at $x = 0$.

We consider the following boundary value problem in Ω :

$$\Delta^2 u = 0 \tag{1.1}$$

$$u\,|_\Gamma = 0 = \frac{\partial u}{\partial n}\Big|_\Gamma \tag{1.2}$$

$$u|_{S_0} = u_{00}, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{S_0} = u_0^{(2)}, \quad u|_{x\to\infty} = 0 \tag{1.3}$$

The conditions (1.3) may be replaced by some other boundary conditions, e.g.:

$$u|_{S_0} = u_0, \quad \left.\frac{\partial u}{\partial x}\right|_{S_0} = u_0^{(1)}, \quad u|_{x\to\infty} = 0 \tag{1.4}$$

It is convenient to consider the problem (1.1) – (1.3) (or (1.1), (1.2) and (1.4)) as a particular case of the following abstract boundary value problem on a semi-axis:

$$u^{(4)}(x) - 2Fu^{(2)}(x) + Vu(x) = 0, \quad u^{(m)} = \partial^m u/\partial x^m. \tag{1.5}$$

$$u(0) = u_0, \ u^{(2)}(0) = u_0^{(2)}, \ u(\infty) = 0 \tag{1.6}$$

or instead of (1.6),

$$u(0) = u_0, \ u^{(1)}(0) = u_0^{(1)}, \ u(\infty) = 0 \tag{1.7}$$

Here $u(x)$ is a function of the variable x in a certain Hilbert space H; V and F are unbounded, positive definite operators, V^{-1} and F^{-1} are completely continuous. Moreover we shall assume that the operator $u = FV^{-1/2}$ is bounded (the fractional powers of the operators are defined in [19]).

When used in connection with the problem (1.1) – (1.3), the operators V and F are denoted by the zero subscript, so that

$$V_0 u = \Delta_0^2 u, \ u|_\gamma = 0 = \left.\frac{\partial u}{\partial n}\right|_\gamma \tag{1.8}$$

$$F_0 u = -\Delta_0 u, \quad u|_\gamma = 0 \left(\Delta_0 = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2}\right).$$

We shall use $L_2(S)$ as H. We define the domains of definition $D(V_0)$ and $D(F_0)$ of the operators V_0 and F_0 as the closure of the set M of functions which are smooth and bounded in S, on the Sobolev spaces $W_2^{(4)}(S)$ and $W_2^{(2)}(S)$, respectively.

2. Summary of the basic definitions.

1°. H_0 is a Hilbert space obtained by the closure of the set M_1 of finite vector functions such that $u(x)$, $u^{(1)}(x)$, $u^{(2)}(x)$, Vu and $Fu^{(1)}$ are continuous in x over H on the metric

$$\|u\|_{H_0}^2 = \int_0^\infty [\|u^{(2)}(x)\|_H^2 + 2\|F^{1/2}u^{(1)}(x)\|_H^2 + \|V^{1/2}u(x)\|_H^2]\,dx \tag{2.1}$$

where $F^{1/2}$ and $V^{1/2}$ are positive roots of the operators F and V, respectively.

H_0' is a Hilbert space obtained by the closure of the set M_1 on the metric

$$\|u\|_{H_0'}^2 = \int_0^\infty [\|u^{(2)}(x)\|_H^2 + \|V^{1/2}u(x)\|_H^2]\,dx \tag{2.2}$$

H^α is the scale of the Hilbert spaces [20] obtained by the closure of the intersection of all $D(V^n)$ $(n = 1, 2, \ldots)$ on the metric

$$\|u\|_{H^\alpha} = \|V^\alpha u\|_H \tag{2.3}$$

When $\alpha < 0$, H^α represents certain spaces of generalized functions. Obviously when $\alpha > \beta$, then $H^\alpha \subset H^\beta$, the imbedding is completely continuous and

$$\| u \|_{H^\alpha} \geqslant \omega \| u \|_{H^\beta}, \quad \omega = \lambda_1^{\alpha-\beta}(V) \tag{2.4}$$

where $\lambda_1(V)$ is the first eigenvalue of the operator V.
$C(H^\alpha)$ is a space of vector functions belonging to H^α, strictly continuous on the ray $x \in [0, \infty)$ and vanishing as $x \to \infty$.

2°. We define the vector function $u^{(n)} \in C(H^\alpha)$ as the nth (continuous) generalized derivative of the vector function $u \in H_0$ provided that a sequence $u_k \in M_1$ exists such, that $\| u_k - u \|_{H_0} \to 0, \quad \max\limits_{0 \leqslant x < \infty} \| u_k^{(n)}(x) - u^{(n)}(x) \|_{H^\alpha} \to 0$

3°. We say that the vector function $u \in H_0$ is a generalized solution of the problem (1.5), (1.6) if it is such, that $u(0) = u_0$ and if it satisfies the identity

$$(u, \psi)_{H_0} = -(\psi^{(1)}(0), u_0^{(2)})_H = l(\psi) \tag{2.5}$$

for any vector function $\psi \in H_0, \psi(0) = 0$. Here $u_0^{(2)}$ must be such, that the functional $l(\psi)$ is continuous in H_0. Below we show that for this requirement it is necessary and sufficient that $u_0^{(2)} \in H^{-1/8}$.

We say that the vector function u is a generalized solution of the problem (1.5), (1.7) if it is such that $u \in H_0$, $u(0) = u_0$, $u^{(1)}(0) = u_0^{(1)}$ and

$$(u, \psi)_{H_0} = 0, \quad \psi \in H_0, \ \psi(0) = \psi^{(1)}(0) = 0 \tag{2.6}$$

for all $\psi \in H_0$ such that $\psi(0) = 0 = \psi^{(1)}(0)$.

4°. We call the elementary solution of (1.5) any of its (generalized) solutions of the form

$$a_k(x) = e^{i\sigma_k x}\left[\frac{x^{p-1}}{(p-1)!}\varphi_{0k} + \frac{x^{p-2}}{(p-2)!}\varphi_{1k} + \ldots + \varphi_{p-1k}\right] \tag{2.7}$$

$$p \geqslant 1, \qquad \operatorname{Im} \sigma_k > 0, \qquad \sigma_k^2 = -\mu_k$$

where μ_k are the eigenvalues and φ_{0k} are the generalized eigenvectors (see [21]) of the operator bundle $\Gamma(\mu)$, i.e.

$$\Gamma(\mu)\varphi \equiv (\mu^2 I + 2\mu F + V)\varphi = 0$$

and φ_{sk} are the adjoint vectors defined by the relations

$$\Gamma(\mu_k)\varphi_{sk} + \frac{\partial \Gamma(\mu_k)}{\partial \mu_k}\varphi_{s-1k} + \frac{1}{2}\frac{\partial^2 \Gamma(\mu_k)}{\partial \mu_k^2}\varphi_{s-2k} = 0$$

3. Several results follow concerning the properties of continuity of the elements belonging to H_0'. Certain of these results are already known [22] and are given without proof.

Lemma 3.1. Let $u \in H_0'$, then $u(x) \in C(H^{3/8})$, $u^{(1)}(x) \in C(H^{1/8})$ and the following inequalities hold for $x \in [0, \infty)$:

$$\| u(x) \|_{H^{3/8}} \leqslant C \| u \|_{H_0'}, \quad \| u^{(1)}(x) \|_{H^{1/8}} \leqslant C \| u \|_{H_0'} \tag{3.1}$$

Lemma 3.2. Let the operator $U = FV^{-1/2}$ be bounded, i.e. the following inequality hold:

$$\| Uu \| \leqslant N \| u \| \tag{3.2}$$

Then the norms defined by the relations (2.1) and (2.2) are equivalent.

Proof. The equivalence of the norms of H_0 and H_0' means that the following inequalities hold:

$$m_1 \| u \|_{H_0'} \leqslant \| u \|_{H_0} \leqslant m_2 \| u \|_{H_0'} \tag{3.3}$$

It is sufficient to prove the right-hand side inequality, as the left-hand side obviously holds when $m_1 = 1$.

First we note that using the Heintz theorem [19] we have from the inequality (3.2):

$$D(V^\alpha) = D(F^{2\alpha})$$

$$\| F^{2\alpha} v \| \leqslant N^{2\alpha} \| V^\alpha v \|, \qquad v = V^{1/2} u \qquad (0 < \alpha \leqslant 1/2) \tag{3.4}$$

Let us consider the integral

$$\int_0^\infty \| F^{1/2} u^{(1)}(x) \|_H{}^2 \, dx = (F^{3/4} u_0, F^{1/4} u_0^{(1)}) - \int_0^\infty (u^{(2)}(x), Fu(x))_H \, dx \tag{3.5}$$

Applying the Cauchy-Buniakowski inequality and the inequalities (3.1), (3.2), (3.4) we obtain

$$\left| \int_0^\infty (u^{(2)}(x), Fu(x))_H \, dx \right| \leqslant \left(\int_0^\infty \| u^{(2)}(x) \|_H{}^2 \, dx \right)^{1/2} \left(\int_0^\infty \| Fu(x) \|_H{}^2 \, dx \right)^{1/2} \leqslant N \| u \|^2_{H_0'}, \tag{3.6}$$

$$| (F^{3/4} u_0, F^{1/4} u_0^{(1)})_H | \leqslant \| V^{3/8} u_0 \|_H \| V^{1/8} u_0^{(1)} \|_H \leqslant NC^2 \| u \|^2_{H_0'} \tag{3.7}$$

after which (3.3) follows from (3.5) – (3.7) at once, and this proves the lemma.

For the operators V_0 and F_0 defined by (1.8) we have an assertion stronger than (3.2).

L e m m a 3.3. The operator $U_0 = F_0 V_0^{-1/2}$ is isometric. This follows from the identity

$$(V_0 u, u)_H = \| V_0^{1/2} u \|^2_H = \int_S | \Delta_0 u |^2 dS = \| F_0 u \|^2_H, \quad u \in D(V_0)$$

after setting $V_0^{1/2} u = v$.

When investigating the generalized solutions of the boundary value problem (1.5), (1.6), we must construct vector functions which belong to H_0 and satisfy the boundary conditions (1.6). This can be done using the following boundary value problem:

$$v^{(4)}(x) + Vv(x) = 0 \tag{3.8}$$

$$v(0) = u_0, \quad v^{(2)}(0) = u_0^{(2)}, \quad v(\infty) = 0$$

Its solution has the form

$$v(x) = \sum_{k=1}^\infty (b_k \cos \gamma_k x + b_k' \sin \gamma_k x) e^{-\gamma_k x} \theta_k \tag{3.9}$$

$$b_k = (u_0, \theta_k)_H, \qquad b_k' = -{}^1/_2 \, \gamma_k^2 (u_0^{(2)}, \theta_k)_H$$

where θ_k are the eigenvectors and $\lambda_k(V) = 4\gamma_k^4$ are the eigenvalues of the operator V. We also have

$$\| v \|^2_{H_0'} = \frac{7\sqrt{2}}{32} \| V^{3/8} u_0 \|^2_H + \frac{13\sqrt{2}}{32} \| V^{-1/8} u_0^{(2)} \|^2_H + \frac{3\sqrt{2}}{16} (V^{3/8} u_0, V^{-1/8} u_0^{(2)})_H \tag{3.10}$$

Thus, Eq. (3.9) defines the solution $v \in H_0'$ (by virtue of Lemma 3.2 also $v \in H_0$) provided that $u_0 \in H^{3/8}$ and $u_0^{(2)} \in H^{-1/8}$. Further, from (3.10) follows the inequality

$$\| v \|^2_{H_0'} \geqslant \sqrt{2} / 16 (3 \| V^{3/8} u_0 \|^2_H + 2 \| V^{-1/8} u_0^{(2)} \|^2_H) \tag{3.11}$$

Relations (3.10) and (3.11) together yield the following lemma.

Lemma 3.4. For the problem (3.8) to have a solution belonging to H_0', it is necessary and sufficient that $u_0 \in H^{3/8}$ and $u_0^{(2)} \in H^{-1/8}$.

We shall now explain in what sense the generalized solution of (3.8) satisfies the boundary conditions. From Lemma 3.1 it follows that the condition $v(0) = u_0$ holds in the sense of convergence in $H^{3/8}$. From the lemma that follows we deduce that the condition $v^{(2)}(0) = u_0^{(2)}$ holds in the sense of convergence in $H^{-1/8}$.

Lemma 3.5. The generalized solution of the problem (3.8) has an nth generalized derivative $v^{(n)} \in C(H^{\beta})$, $\beta = (3-2n)/8 \quad (n = 0.1...)$.

Proof. When the series (3.9) contains a finite number of terms, the assertion is obvious. The passage to the general case can be made with the help of the Banach-Steinhaus theorem, using the estimate $\| v^{(n)} \|_{H^{\beta}} \leqslant C_n \| v \|_{H_0'}$ derived from the Parseval equation and the expression (3.9) for the generalized solution.

Lemma 3.6. The generalized solution of the problem (1.5),(1.6) has an nth generalized derivative $u^{(n)} \in C(H^{\beta})$, $\beta = (3-2n)/8$ $(n = 0, 1, 2, 3)$.

This lemma is proved in exactly the same manner as Lemma 3.5 (also see the proof of Lemma 2.2 in [23]).

4. Let us formulate the theorems of existence of solutions of the boundary value problems (1.5),(1.6) and (1.5),(1.7).

Theorem 4.1. For the problem (1.5),(1.6) to have a unique solution in the space H_0, it is necessary and sufficient that $u_0 \in H^{3/8}$ and $u_0^{(2)} \in H^{-1/8}$.

Proof. Let u be a generalized solution of the problem (1.5),(1.6). By the Riesz theorem and by Lemma 3.2 there exists a vector function $v \in \mathbf{H}_0'$ satisfying the identity

$$(v, \psi)_{H_0'} = (u, \psi)_{H_0} = l(\psi) \tag{4.1}$$

which follows from the definition (2.5). Here v is the generalized solution of the problem (3.8). The necessity now follows from Lemma 3.4. To prove the sufficiency, we shall seek the generalized solution in the form $u = v + w$, where v is the generalized solution of the boundary value problem (3.8). Then by (2.5) the vector function w must satisfy the following integral identity:

$$(w, \psi)_{H_0} = -2\int_0^{\infty} (F^{1/2} v^{(1)}(x), \quad F^{1/2}\psi^{(1)}(x))_H dx, \quad \psi \in H_0 \tag{4.2}$$

and the condition $w(0) = 0$. The right-hand side of (4.2) defines a functional, continuous in ψ on the subspace $H_{00} = \{\psi \in H_0, \ \psi(0) = 0\}$, therefore the vector function $w \in H_0$ exists and is unique.

Theorem 4.2. For the problem (1.5),(1.7) to have a unique solution in the space H_0, it is necessary and sufficient that $u_0 \in H^{3/8}$ and $u_0^{(1)} \in H^{1/8}$.

The proof of this theorem is analogous to that of Theorem 4.1 and is somewhat simpler, since it is based on Lemmas 3.1 and 3.2 only, with no need to use further properties of (1.5).

5. Using the results of [21] we shall prove the double completeness of the Jordan arrays in (2.7), assuming that the operators V and F are defined by the following relations:

$$V\varphi = \alpha \frac{d^4\varphi}{dy^4}, \qquad \varphi|_{y=\pm 1} = 0 = \frac{d\varphi}{dy}\Big|_{y=\pm 1} \tag{5.1}$$

$$F\varphi = -\beta \frac{d^2\varphi}{dy^2}, \qquad \varphi|_{y=\pm 1} = 0, \;\; \alpha, \; \beta > 0, \qquad y \in S = [-1, 1] \tag{5.2}$$

In particular we can reduce to Eq. (1.5) with the operators V and F of the type (5.1) and (5.2), the problem of bending an orthotropic plate clamped rigidly along two opposite sides and the plane problem of the theory of elasticity for an orthotropic material. The case $\alpha = \beta = 1$ corresponds to an isotropic material.

Following [21], we make the following substitution in (2.7):

$$\psi = V^{1/2}\varphi, \;\; \mu^{-1} = \lambda \tag{5.3}$$

and pass to the associate equation

$$L(\lambda)\psi \equiv (\lambda^2 I + 2\lambda B + C)\psi = 0$$

$$(C = V^{-1}, \; B = C^{1/2} F C^{1/2}, \; \psi \in H)$$

It is clear that C and B are positive, completely continuous operators. We say that the sequence $\psi_{k0}, \psi_{k1}, \ldots, \psi_{kp-1}$, where

$$L(\lambda_k)\psi_{k0} = 0, \;\; L(\lambda_k)\psi_{kj} + \frac{\partial L(\lambda_k)}{\partial \lambda_k}\psi_{kj-1} + \frac{1}{2}\frac{\partial^2 L(\lambda_k)}{\partial \lambda_k^2}\psi_{kj-2} = 0$$

forms a Jordan array of the bundle $L(\lambda)$ corresponding to the eigenvalue λ_k. We denote by $\{\psi_k\}$ the set of all eigenvectors and adjoint vectors of the bundle $L(\lambda)$. Following [21] we say that the system of Jordan arrays $\{\psi_k\}$ is doubly complete if for any two functions $f, g \in H$ and $\delta > 0$ such n and such constants $C_k^{(n)}$ can be found that the following inequalities hold simultaneously:

$$\left\| f - \sum_{k=1}^{n} \lambda_k^{-1} C_k^{(n)} V^{-1/2} \psi_k \right\|_H < \delta, \quad \left\| g - \sum_{k=1}^{n} C_k^{(n)} \psi_k \right\|_H < \delta \tag{5.4}$$

According to Krein ([21], theorem 2.1, p. 297) the double completeness occurs each time when $\lim n^2 \lambda_n(C) = 0$, where $\lambda_n(C)$ are the eigenvalues of the operator C. Since $C = V^{-1}$, we have $\lambda_n(C) = \lambda_n^{-1}(V)$. The asymptotics of the eigenvalues of the boundary value problem [24] for the operator V defined by the relations (5.1) has the form $\lambda_n(V) = (2\pi n)^4 [1 + O(1/n)]$, consequently the condition given above holds.

From the double completeness which we have proved follow the inequalities (5.4) or, after making the substitutions (5.3), the inequalities

$$\left\| f - \sum_{k=1}^{n} \mu_k C_k^{(n)} \varphi_k \right\|_H < \delta, \quad \left\| g - \sum_{k=1}^{n} C_k^{(n)} V^{1/2} \varphi_k \right\|_H < \delta \tag{5.5}$$

Here $\{\varphi_k\}$ is the complete system of eigenvectors and adjoint vectors of the bundle $\Gamma(\mu)$.

6. Let us consider the problem of completeness of the elementary solutions (2.5) in the space of solutions of (1.5) belonging to H_0. We denote this space by H_{0r} and prove the following theorem.

Theorem 6.1. For any function $u \in H_{0r}$ and $\varepsilon > 0$ a function

$$u_n = C_1^{(n)} a_1(x) + \ldots + C_n^{(n)} a_n(x)$$

can be found such, that

$$\| u - u_n \|_{H_0} < \varepsilon \tag{6.1}$$

Proof. We use Theorem 4.1 to assert that the following inequality holds:

$$\| u - u_n \|_{H_0} \leqslant \omega_1 \left[\| u_0 - u_n(0) \|_{H^{3/8}} + \| u_0^{(2)} - u_n^{(2)}(0) \|_{H^{-1/8}} \right] \tag{6.2}$$

Further, setting in (5.5) $f = u_0^{(2)}$ and $g = V^{1/2} u_0$ we find

$$\| u_0 - u_n(0) \|_{H^{1/2}} < \delta, \quad \| u_0^{(2)} - u_n^{(2)}(0) \|_H < \delta \tag{6.3}$$

From the inequality (2.4) we have

$$\| u_0 - u_n(0) \|_{H^{3/8}} \leqslant \omega_2 \| u_0 - u_n(0) \|_{H^{1/2}} < \omega_2 \delta \tag{6.4}$$

$$\| u_0^{(2)} - u_n^{(2)}(0) \|_{H^{-1/8}} \leqslant \omega_2 \| u_0^{(2)} - u_n^{(2)}(0) \|_H < \omega_2 \delta$$

$$\omega_2 = \lambda_2^{-1/4}(V)$$

Inserting (6.4) into (6.2) and setting $2\omega_1\omega_2\delta = \varepsilon$, we obtain the inequality (6.1).

From Theorem 4.1 follow, in particular, the solvability of an infinite system obtained by minimizing the functional

$$\left\| u - \sum_{k=1}^{n} C_k^{(n)} a_k(x) \right\|_{H_0}^2$$

over the constants $C_k^{(n)}$, the convergence of the method of reduction and the convergence of the approximate solutions u_n, with $n \to \infty$, to the generalized solution of the biharmonic equation. It can be shown that the infinite system obtained in this way is equivalent to the system given in [3, 4] obtained on the basis of the asymptotic method and the Lagrange's variational principle.

7. In conclusion we consider a semi-strip the elastic properties of which vary across its thickness. In this case the operators V and F become

$$Vu = \frac{1}{p}\frac{d^2}{dy^2}\left(p\frac{d^2u}{dy^2}\right), \quad u\Big|_{y=\pm 1} = 0 = \frac{du}{dy}\Big|_{y=\pm 1} \tag{7.1}$$

$$Fu = -\frac{1}{2p}\left[\frac{d}{dy}\left(p_1\frac{du}{dy}\right) - \frac{d^2}{dy^2}(p_2 u) - p_2\frac{d^2u}{dy^2}\right], \quad u\,\Big|_{y=\pm 1} = 0$$

$$p = \frac{1-\nu}{2\mu}, \qquad p_1 = \frac{1}{2\mu}, \qquad p_2 = \frac{\nu}{2\mu}$$

Here $\mu = \mu(y)$ is the shear modulus and $\nu = \nu(y)$ the Poisson's ratio, respectively. Since $\mu > 0$ and $0 < \nu < 1/2$, the functions p, p_1 and p_2 can be assumed positive on the segment $y \in [-1, 1]$. For the space H we use $L_2(S)$ with weight p. We assume that the functions p and p_2 are twice continuously differentiable and p_1 is continuously differentiable on the segment $y \in [-1, 1]$. In the present case the operator F is not, generally speaking, positive, but it can be shown that it is bounded from below, i.e.

$$(Fu, u)_H \geqslant k\,(u, u)_H$$

The substitution $\mu \to \mu + a$ in the bundle (2.7) yields another bundle of the same type in which $F \to F + aI$ and $V \to V + 2aF + a^2 I$. Clearly when a are large and positive, we arrive back at an operator equation of the form (2.7) with positive F and V. Under these conditions the operators V^{-1} and F^{-1} satisfy all conditions necessary for the application of the basic theorem of Krein [21], therefore the conclusions of

Theorem 6.1 remain valid for the problems (7.1).

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INVESTIGATION OF THE ALGEBRAIC SYSTEM OF INFINITE ORDER OCCURRING IN SOLVING THE PROBLEM FOR A SEMI-STRIP

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The algebraic system of equations of infinite order studied here occurs during the solution of the problem of the theory of elasticity concerning a symmetrically loaded semi-strip clamped at the one end. The system is solved using the iteration method. First, out of the matrix of the system a sub-matrix is selected, characterizing the behavior of the solution at large values of the index of the unknown. It is proved and confirmed by concrete examples, that the solution of the basic system differs little from the solution of a simplified system. An asymptotic expansion is obtained for the solution of the simplified system for the large values of the index of the unknown and an approximate method is given for the determination of its coefficients.

An infinite system of algebraic equations for a semi-strip with stress-free longitudinal edges and displacements specified at its end was discussed in [1] where it was proved that the system is completely regular. Earlier [2] the behavior of the solution at large values of the index of the unknown was explained